Assignment 5

Hand in no. 4, 6, and 7 by October 28, 2024.

1. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

2. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting

$$d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right| .$$

   (a) Show that it is not a complete metric.

   (b) Describe how to make it complete by adding one new point.

3. Optional. Let (X, d) be a metric space. Fixing a point $p \in X$, for each x define a function

$$f_x(z) = d(z,x) - d(z,p).$$

   (a) Show that each f_x is a bounded, uniformly continuous function in X.

   (b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$. In other words,

$$\|f_x - f_y\|_\infty = d(x,y), \quad \forall x, y \in X.$$

   (c) Deduce from (b) the completion theorem asserting that every metric space has a completion.

   This approach is shorter than the proof given our notes. However, it is not so inspiring.

4. Let T be a continuous map on the complete metric space X. Suppose that for some k, T^k becomes a contraction. Show that T admits a unique fixed point. This generalizes the contraction mapping principle in the case $k = 1$.

5. Show that the equation $x = \dfrac{1}{2}\cos^2 x$ has a unique solution in $\mathbb{R}$.

6. Show that the equation $2x \sin x - x^4 + x = 0.001$ has a root near $x = 0$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists some $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$

   is a contraction. This provides a justification for Newton's method in finding roots for an equation.